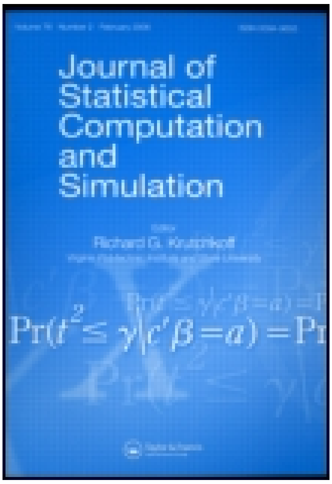

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A modified runs test for symmetry

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A modified runs test for symmetry

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We propose a modification of a Modarres–Gastwirth test for the hypothesis of symmetry about a known center. By means of a Monte Carlo Study we show that the modified test overtakes the original Modarres–Gastwirth test for a wide spectrum of asymmetrical alternatives coming from the lambda family and for all assayed sample sizes. We also show that our test is the best runs test among the runs tests we have compared.

Keywords: runs test; test of symmetry; generalized lambda family; power

AMS Subject Classification: Primary 62G10; Secondary 62G30

1. Introduction

Let $X_1, \dots, X_n$ be a sequence of independent random variables with distribution function F, density f and known median equal to zero. Consider the test problem:

$$H_0 : F(x) = 1 - F(-x)$$

which indicates that F is symmetic about zero, against the asymmetric alternative:

$$K_1 : F(x) \neq 1 - F(-x).$$

This test problem has received considerable attention in the literature; see, for example [1–8].

In Section 2, we introduce the proposed test and give some distributional properties of the test statistic. Section 3 shows the results of a simulation study to compare our test with other tests. Section 4 contains some conclusions and a discussion of the main results.

2. The modified test

Let $|X|_{(1)}, \dots, |X|_{(n)}$ be the sequence of ordered absolute values of the observations. Define the antirank D_j of $|X|_{(j)}$ by $|X_{D_j}| = |X|_{(j)}, j = 1, \dots, n$, and let

$$S(X_{D_j}) = \begin{cases} 1 & \text{if } X_{D_j} > 0, \\ 0 & \text{otherwise.} \end{cases} \quad j = 1, \dots, n. \tag{1}$$

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To avoid notational complexity, we use only S_j to denote the ordered sequence of ones and zeros; i.e. $S(X_{D_j}) = S_j$, although S depends on X and D_j.

Let

$$I_j = \begin{cases} 1 & \text{if } S_{j-1} \neq S_j, \\ 0 & \text{if } S_{j-1} = S_j \end{cases} \quad j = 2, \dots, n,$$

be a counter of the number of changes in the dichotomized sequence. Define the test statistic by

$$J_k = 1 + \sum_{i=1}^{k} (n - k + i) I_{n-k+i}, \quad k = 2, \dots, n-1 \tag{2}$$

which corresponds to the number of runs in the last k terms of the sequence $I_2, \dots, I_n$ weighted according to their positions.

A critical region for the J_k test can be built following the argument of McWilliams [1], which establishes that under asymmetry of F and, for $b > a > 0$, either $P(a < X < b) > P(-b < X < -a)$, which means $P(X > 0) > P(X < 0)$ or $P(a < X < b) < P(-b < X < -a)$, which means $P(X > 0) < P(X < 0)$. Therefore, by definition (1), clusters will be expected at the ends of the dichotomized sequence (clusters of ones when $P(X > 0) > P(X < 0)$ holds, and cluster of zeros when $P(X > 0) < P(X < 0)$ holds). So the tails of the dichotomized sample contain information about the tails of the sample distribution. The foregoing reasoning leads to rejecting the null hypothesis of symmetry for small values of J_k.

Since $X_{D_1}, \dots, X_{D_n}$ is a permutation of the original independent random variables the random variables $S_1, \dots, S_n$ are also independent with Bernoulli distributed with parameter $p = P(X_{D_j} > 0) = \frac{1}{2}$ under H_0, and from the last independence follows that $I_2, \dots, I_n$ are also Bermoulli independent distributed with the same parameter.

THEOREM 2.1 *Under the null hypothesis of symmetry*, J_k *has mean and variance*

$$E(J_k - 1) = \frac{1}{2} \sum_{i=1}^{k} (n - k + i) = \frac{1}{4} k(2n - k + 1),$$

$$V(J_k - 1) = \frac{1}{4} \sum_{i=1}^{k} (n - k + i)^2 = \frac{1}{24} k(6n^2 + 6n + 2k^2 - 3k - 6nk + 1)$$

THEOREM 2.2 *Let* $0 < \gamma < 1$ *and* $k_n = [\gamma n]$ *such that* $k_n \to \infty$ *when* $n \to \infty$. *By replacing* k *by* k_n *in* (2) *an expresion for* J_k *depending on n is obtained. Then*

$$\frac{(J_{k_n} - 1) - E(J_{k_n} - 1)}{\sqrt{V(J_{k_n} - 1)}}$$

converges to the standard normal distribution.

Proof The test statistic is a linear combination of independent random variables and therefore is enough to prove the Noether contition:

$$\frac{\max_{1 \le i \le k_n} (n - k_n + i)}{\sqrt{\sum_{i=1}^{k_n} (n - k_n + i)^2}} = \frac{n}{\sqrt{(k_n/6)(2k_n^2 - 3k_n(2n+1) + 6n^2 + 6n + 1)}} \xrightarrow{n \to \infty} 0.$$

■

3. Monte Carlo study

This section contents the results of a Monte Carlo Study to compare the J_k test with the following tests:

1. The McWilliams test based on the total number of runs [1]:

$$R^* - 1 = \sum_{i=2}^{n} I_i.$$

2. The Baklizi test based on a modified runs test conditioned on the number of observations greater than the specified median, R^* given n_1 and n_2, the numbers of ones and zeros in the $S_1, \ldots, S_n$ sequence [5]. This is the $R^*/n_1, n_2$ test.
3. The Wilcoxon Signed-Rank Test suggested by [9] as a test for symmetry:

$$W = \sum_{i=1}^{n} iS_i.$$

4. The Tajjudin test, based on the Wilcoxon two sample test commonly used for location [2]

$$T = \sum_{i=1}^{n} iS_i = \sum_{i=1}^{n_1} R_i,$$

where R_i are the ranks of the n_1 positive observations in the sample.
5. The Cheng–Balakrishnan C_6 test, which is based on the sum of the last six terms in the sequence $S_1, \ldots, S_n$ [6].
6. The Modarres M_p test [3]:

$$M_p = \sum_{i=[np]+2}^{n} \varphi(i) I_i,$$

where

$$\varphi(i) = \begin{cases} i - np & \text{if} i > np, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $k = n - 1$, the expresion for $J_k - 1$ in (2) is equivalent to M_0.
7. The Baklizi [7] test denoted by $L^*_{n,p}$, which uses the longest run in the signs of the largest $n - [np]$ trials in the sequence $S_1, \ldots, S_n$, where $[np]$ is the integer part of np, and p is a trimming proportion. $L^* = L^*_{n,0}$ denotes the length of the longest run of either type.
8. The second, Baklizi [7] test. Let μ_p and σ_p^2 be the mean and variance of $L^*_{n,p}$ and define

$$L = \frac{\sup_p (L^*_{n,p} - \mu_p)}{\sigma_p}.$$

9. The Baklizi test [8] denoted by R_{modif} corresponds to the two stage hybrid test [4] without the first stage, which does not contribute to the power.

Following [1–8], we simulated samples of sizes $n = 20, 30, 50, 100$ from nine cases of the generalized lambda distribution (GLD). So the generated data were obtained as $x_i^* = \lambda_1 + (u_i^{\lambda_3} - (1 - u_i)^{\lambda_4})/\lambda_2,\ i = 1, \ldots, n$, where u is a uniform random number. The cases of the GLD were chosen as in McWilliams [1], and these were reordered such that the power was increasing. The first case corresponds to the normal distribution approximated by the GLD and plays the role of the null hypothesis to calibrate the size of the compared tests. The values of the four parameters for the nine selected densities of the GLD are listed in Table A1 and are plotted in Figure A1.

Table A2 shows the proportion of rejections of the null hypothesis for all compared tests, based on 10, 000 replications with $\alpha = 0.05$. The test $R^* - 1$ was randomized to achieve the nominal size 0.05. To select the value of k in the J_k test, we calculated the power of J_4 to J_{20} used in Monte Carlo simulations. The comparisons showed that J_6 has the best power properties (Table A3).

For all sample sizes in case 1 (symmetric distribution), all tests compared maintain the nominal size around 0.05. For sample size $n = 20$ and for cases 2–5, the greater powers are reached by the J_6 test and the R_{modif} test. In cases 6–9, the J_6 test clearly dominates its competitors. For sample size $n = 30$ in case 2 all tests have similar behaviour. In case 3, the R_{modif} test is the best behaved, closely followed by the $L^*_{n,0.8}$ and J_6 tests. In cases 4 and 5, the four best behaved are the R_{modif} and J_6 tests. In cases 6–9, the J_6 test is the best behaved, followed by the R_{modif}, $M_{.60}$ and $L^*_{n,0.8}$.

For sample size $n = 50$ in cases 2–5 the highest powers are obtained by the R_{modif}, J_6 and C_6 tests. In cases 6–9, the J_6 and C_6 tests have the largest powers.

For sample $n = 100$, the highest powers are reached by the R_{modif}, J_6 and C_6 tests.

4. Conclusions and discussion

Results show that the J_6 test maintains its nominal size for all sample sizes. Our test competes well with the best-known symmetry runs tests, in all chosen GLD cases and for the sample sizes used in this work.

Our test is a modification of the Modarres test [3], and is more powerful than the original M_p test in all cases and for all assayed sample sizes. The only test that overtakes our test is the rank test R_{modif}. This means that the J_6 test is, up to now, the best.

We have included the W test because Gibbons and Chakraborti [9] suggest the use of the Wilcoxon Signed-Rank statistic to test the hypothesis of symmetry. However, our results indicate that this test has a poor power for this aim.

We have supposed that the median of the sampled distribution is known, but in practice this is an unrealistic assumption. To explore how robust is the proposed test to this assumption we simulated samples from distributions with unknown medians for the same cases of the GLD and the same sample sizes used in this work. We found that the sizes and the powers of the J test remain the same as for the cases with known medians. One possible reason for this is that the test statistic uses only the six largest observations and hence the J-test tolerates changes in the $n - 6$ smallest observations before it changes the decision about the rejection or acceptation of the null hypothesis. If we use the sample median as estimator of the true median, it is unlikely, for large values of n, to expect modifications in the largest observations.

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Appendix 1.

Table A1. Values of the four parameters for the nine selected cases of the GLD used for the Monte Carlo study.

Case	λ_1	λ_2	λ_3	λ_4	α_3	α_4
1 (H_0)	0.000000	0.197454	0.134915	0.134915	0.0000	3.0000
2	−0.116734	−0.351663	−0.130000	−0.160000	0.8000	11.4000
3	0.000000	−1.000000	−0.100000	−0.180000	2.0000	21.2000
4	3.586508	0.043060	0.025213	0.094029	0.9000	4.2000
5	0.000000	−1.000000	−0.007500	−0.030000	1.5000	7.5000
6	0.000000	1.000000	1.400000	0.250000	0.5000	2.2000
7	0.000000	1.000000	0.000070	0.100000	1.5000	5.8000
8	0.000000	−1.000000	−0.001000	−0.130000	3.1600	23.8000
9	0.000000	−1.000000	−0.000100	−0.170000	3.8800	40.7000

Table A2. Empirical power

Distribution	n	$R^* - 1$	$R^*/n_1.n_2$	W	T	C_6	$M_{.20}$	$M_{.25}$	$M_{.60}$	J_6
	20	0.048	0.052	0.046	0.046	0.047	0.046	0.047	0.045	0.056
Case 1 (H_0)	30	0.050	0.053	0.052	0.050	0.052	0.050	0.051	0.054	0.047
	50	0.050	0.054	0.050	0.052	0.050	0.050	0.049	0.049	0.046
	100	0.051	0.047	0.052	0.048	0.052	0.053	0.053	0.051	0.048
	20	0.052	0.057	0.050	0.051	0.054	0.053	0.054	0.053	0.070
Case 2	30	0.052	0.051	0.055	0.056	0.061	0.053	0.053	0.055	0.061
	50	0.055	0.056	0.052	0.060	0.070	0.063	0.062	0.066	0.068
	100	0.054	0.051	0.055	0.071	0.091	0.057	0.057	0.062	0.084
	20	0.067	0.075	0.055	0.079	0.080	0.077	0.079	0.087	0.112
Case 3	30	0.074	0.075	0.062	0.097	0.119	0.092	0.094	0.109	0.125
	50	0.089	0.094	0.064	0.131	0.204	0.121	0.120	0.153	0.206
	100	0.113	0.109	0.088	0.224	0.366	0.169	0.169	0.217	0.356
	20	0.090	0.103	0.061	0.106	0.118	0.120	0.122	0.142	0.177
Case 4	30	0.114	0.122	0.070	0.149	0.219	0.163	0.166	0.199	0.243
	50	0.143	0.154	0.085	0.209	0.428	0.231	0.234	0.301	0.443
	100	0.216	0.209	0.127	0.385	0.757	0.398	0.406	0.522	0.750
	20	0.115	0.131	0.067	0.133	0.155	0.158	0.162	0.190	0.235
Case 5	30	0.151	0.160	0.080	0.194	0.309	0.228	0.232	0.287	0.343
	50	0.197	0.213	0.103	0.287	0.587	0.338	0.342	0.437	0.602
	100	0.321	0.316	0.166	0.522	0.890	0.555	0.566	0.696	0.885
	20	0.200	0.234	0.072	0.160	0.256	0.334	0.346	0.396	0.468
Case 6	30	0.303	0.330	0.095	0.231	0.606	0.546	0.558	0.671	0.715
	50	0.497	0.524	0.122	0.364	0.950	0.815	0.825	0.920	0.972
	100	0.782	0.782	0.198	0.633	1.000	0.986	0.989	0.998	1.000
	20	0.311	0.358	0.096	0.281	0.421	0.496	0.511	0.578	0.644
Case 7	30	0.457	0.490	0.123	0.393	0.797	0.737	0.750	0.828	0.868
	50	0.683	0.707	0.185	0.600	0.991	0.937	0.941	0.977	0.994
	100	0.928	0.927	0.358	0.883	1.000	0.999	0.999	1.000	1.000
	20	0.373	0.426	0.105	0.330	0.494	0.581	0.594	0.656	0.715
Case 8	30	0.539	0.570	0.150	0.484	0.861	0.810	0.819	0.878	0.915
	50	0.761	0.782	0.233	0.697	0.996	0.967	0.970	0.989	0.998
	100	0.966	0.965	0.420	0.947	1.000	1.000	1.000	1.000	1.000
	20	0.399	0.452	0.112	0.351	0.530	0.618	0.631	0.696	0.752
Case 9	30	0.580	0.614	0.152	0.498	0.877	0.841	0.848	0.900	0.929
	50	0.802	0.821	0.241	0.725	0.997	0.977	0.979	0.992	0.999
	100	0.980	0.980	0.441	0.956	1.000	1.000	1.000	1.000	1.000

$\alpha = 0.05$ with 10,000 replications.

Table A3. Empirical power.

Distribution	n	L^*	$L^*_{n.0.8}$	L	R_{Modif}	J_6
	20	0.050	0.055	0.055	0.045	0.056
Case 1 (H_0)	30	0.051	0.054	0.048	0.045	0.047
	50	0.049	0.050	0.058	0.050	0.046
	100	0.055	0.054	0.049	0.051	0.048
	20	0.057	0.062	0.046	0.058	0.070
Case 2	30	0.050	0.063	0.058	0.062	0.061
	50	0.058	0.062	0.053	0.075	0.068
	100	0.053	0.066	0.065	0.106	0.084
	20	0.057	0.088	0.070	0.114	0.112
Case 3	30	0.069	0.128	0.088	0.156	0.125
	50	0.075	0.145	0.141	0.253	0.206
	100	0.122	0.228	0.233	0.486	0.356
	20	0.072	0.138	0.087	0.187	0.177
Case 4	30	0.100	0.229	0.142	0.287	0.243
	50	0.144	0.303	0.314	0.499	0.443
	100	0.333	0.572	0.595	0.818	0.750
	20	0.095	0.165	0.120	0.254	0.235
Case 5	30	0.131	0.333	0.219	0.404	0.343
	50	0.230	0.457	0.455	0.668	0.602
	100	0.556	0.769	0.784	0.939	0.885
	20	0.136	0.267	0.191	0.420	0.468
Case 6	30	0.256	0.649	0.469	0.653	0.715
	50	0.642	0.908	0.914	0.894	0.972
	100	0.995	1.000	1.000	0.994	1.000
	20	0.226	0.330	0.314	0.593	0.644
Case 7	30	0.444	0.823	0.689	0.854	0.868
	50	0.860	0.978	0.980	0.980	0.994
	100	1.000	1.000	1.000	1.000	1.000
	20	0.295	0.366	0.389	0.666	0.715
Case 8	30	0.555	0.876	0.790	0.913	0.915
	50	0.930	0.991	0.991	0.993	0.998
	100	1.000	1.000	1.000	1.000	1.000
	20	0.322	0.359	0.428	0.692	0.752
Case 9	30	0.608	0.898	0.821	0.924	0.929
	50	0.953	0.993	0.995	0.995	0.999
	100	1.000	1.000	1.000	1.000	1.000

$\alpha = 0.05$[a].
[a]Taken from [7,8].

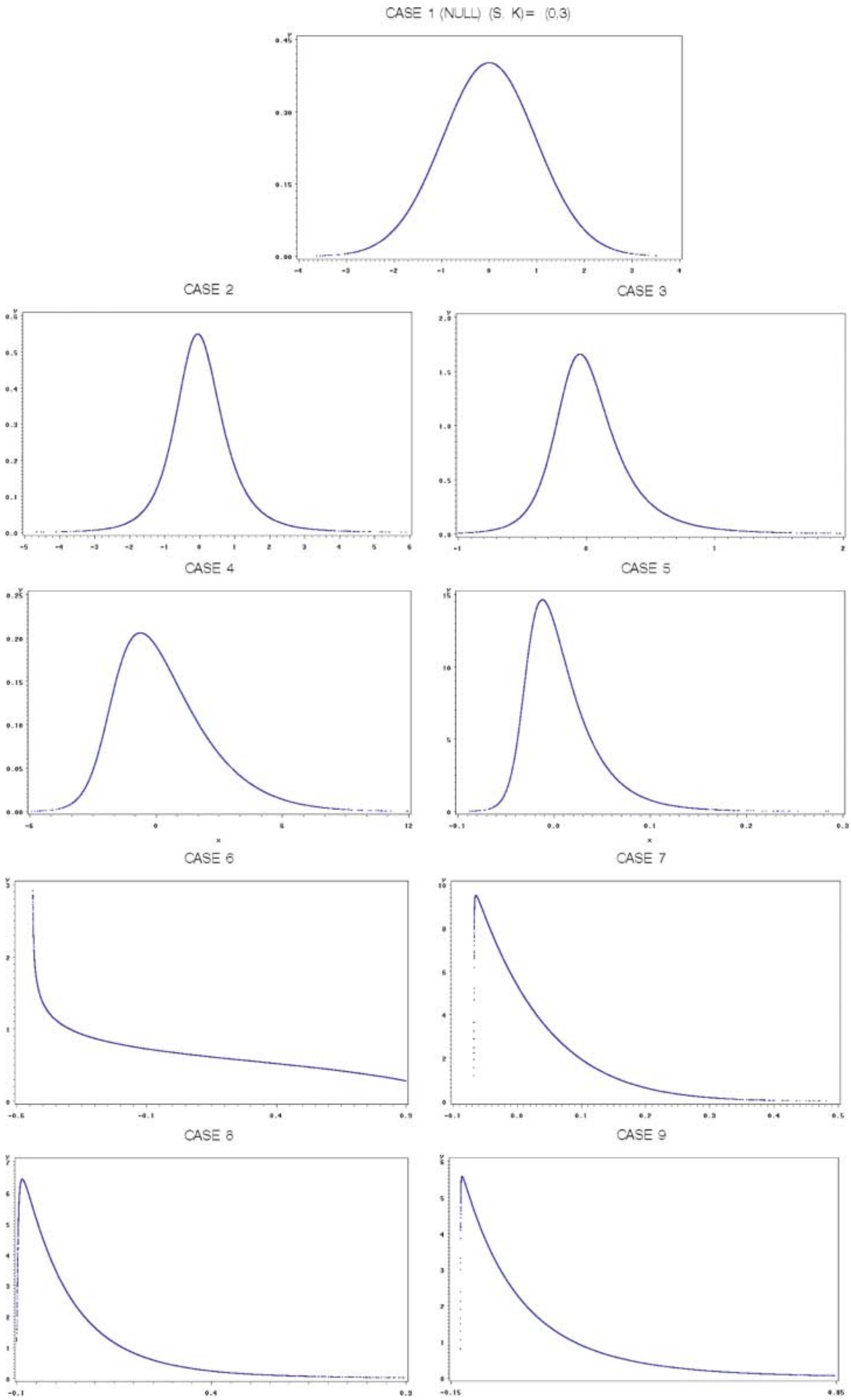

Figure A1. Density functions for selected cases of the GLD.