Analytical study of the magnetic field generated by multipolar magnetic configuration



2016 J. Phys.: Conf. Ser. 687 012023

(http://iopscience.iop.org/1742-6596/687/1/012023)

# Analytical study of the magnetic field generated by multipolar magnetic configuration

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**Abstract.** The magneto-statics field from a parallelepiped magnet which can turn around an axis, is the first step to find the whole magnetic field in a multipolar configuration. This configuration is present in the ion sources, which are heated by electron cyclotron resonance. We present the analytic formulas to calculate this magnetic field outside the volume of the magnet. To model the magnet, we considered a constant magnetization vector inside of magnet volume. Therefore, the magnetic scalar potential method can be used. We present the results by a hexapolar system. Their magnetic field components are calculated on confinement region, several graphics are shown with directions and magnitude's gradients of the magnetic field to help understand better the confinement system. Our results are confronted with experimental ones. These formulas are very useful in research of plasma magnetic confinement in ion sources through computational simulations.

## 1. Introduction

To confine plasma in an electron cyclotron resonance (ECR) ion source, a transversal multicusp magnetic field is used. This field helps to remove magnetohydrodynamics instabilities due to the convex curvature of mirror's magnetic field [1]. It is common to use six or eight parallelepiped magnets around cylinder discharge camera to create a cusp geometric form of magnetic field, which changes the magnetic field curvature. The analytic calculation of magnetic field is useful in plasma dynamics computational simulations. Although several models use multipolar approximation, the interaction between the magneto-statics field of the trap and the microwaves field exerts the biggest influence over plasma behaviour; therefore is very important to get a better model to calculate magnetic field. An analytical result was published in [4] for the modelling of parallelepipedic magnets of various polarisation directions. However, we present the case, when the polarisation vector stays constant, but the whole magnet can turn around an axis and we solve each indetermination present in the formulas. Therefore, our formulas are more useful for calculate multipolar magnetic field. The multipolar magnetic field has been important in plasma studying from microelectronics fabrication to the fabrication flat panel display device [2]. This system of magnetic field is used in some configurations of magnetic confinement [3]. It starts from a magnetic scalar potential [5], but its equations are not solved, instead the magnetic fields components are calculated by using gradients, transforming those equations until they transform into a integrable form. The final equations are using to calculate the magnetic field into a cubic mesh. The magnet is considered as a material with constant magnetisation vector inside it, and zero outside. The equations are solved only in confinement volume, which is found outside magnets inner region. Several pictures show the curvature and magnitudes gradients.

## 2. Magnet modelling

The magnet is modelling by considering a constant magnetisation vector inside their volume, it is oriented radially from a pole toward the opposite pole, as is showed in Figure 1. Since the interest region is found outside magnet, the magnetic field can be calculated through the method of gradient of scalar magnetic potential [5] since, not conduction density current exists in this place: $\phi_m(\mathbf{r}) = \int_{s_{(v)}} \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \cdot d\mathbf{s}' - \int_v \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dv'$.Where, $\phi_m$ is the scalar magnetic potential, $s_{(v)}$ is the magnet boundary surface, $v$ is the magnet volume. However, the second term is zero since the vector $\mathbf{M}$ is constant inside. Then the potential equation can be changed by:

$$\phi_m(\mathbf{r}) = \int_{z_1}^{z_2} \int_{x_2}^{x_{2R}} \frac{dx'dz'}{|\sin\theta|} \frac{M_0}{|\mathbf{r}-\mathbf{r}'|} - \int_{z_1}^{z_2} \int_{x_1}^{x_{1R}} \frac{dx'dz'}{|\sin\theta|} \frac{M_0}{|\mathbf{r}-\mathbf{r}'|} \tag{1}$$

Where $x_{2R}$, $x_2$, $x_{1R}$ and $x_1$ are the limits on $X$ axis, as is showed in Figure 1. The $z_1$ and $z_2$ are the limits on $z$ axis. The limits to $x$ axis can be found in this way: $x_1 = R_1 \cos\theta - \frac{a}{2}|\sin\theta|$ and $x_{1R} = R_1 \cos\theta + \frac{a}{2}|\sin\theta|$ for inner pole's top. $x_2 = R_2 \cos\theta - \frac{a}{2}|\sin\theta|$ and $x_{2R} = R_2 \cos\theta + \frac{a}{2}|\sin\theta|$ for outer pole's top.

We use the follow notation $\phi(x_l, x_r, \theta, R)(\mathbf{r}) = \int_{z_1}^{z_2} \int_{x_l}^{x_r} \frac{dx'dz'}{|\sin\theta|} \frac{M_0}{|\mathbf{r}-\mathbf{r}'|}$, therefore: $\phi_m(\mathbf{r}) = \phi(x_2, x_{2R}, \theta, R_2)(\mathbf{r}) - \phi(x_1, x_{1R}, \theta, R_1)(\mathbf{r})$. The integral respect to $z$ can be solved, so we get:

$\phi(x_l, x_r, \theta, R)(\mathbf{r}) = f(x_l, x_r, \theta, R, z_2)(\mathbf{r}) - f(x_l, x_r, \theta, R, z_1)(\mathbf{r})$, where: $f(x_l, x_r, \theta, R, z_b)(\mathbf{r}) = -\frac{M_0}{\sin\theta} \int_{x_l}^{x_r} \ln(\sqrt{(z-z_b)^2 + \rho^2(\theta, R, x')} + z - z_b)\, dx'$ and $\rho^2(\theta, R, x') = (x-x')^2 + \left(y - \frac{R}{\sin\theta} + x' \cot\theta\right)^2$.

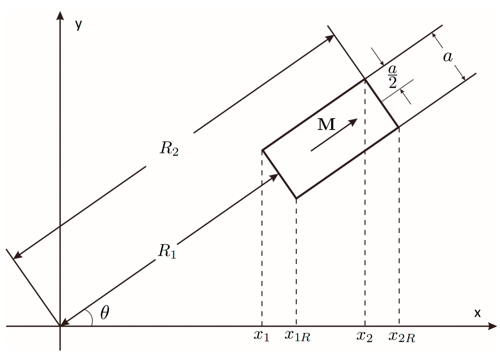

**Figure 1.** Reference system (Magnet bar cross-section).

### $B_x$ component

Since $B_x = -\frac{\partial \phi_m}{\partial x}$ we need to solve:

$$\frac{\partial f(x_l, x_r, \theta, R, z_b)(\mathbf{r})}{\partial x} = -\frac{M_0}{|\sin\theta|} \int_{x_l}^{x_r} \frac{(x-x')}{\sqrt{(z-z_b)^2 + \rho^2(\theta, R, x')} + z - z_b} \frac{dx'}{\sqrt{(z-z_b)^2 + \rho^2(\theta, R, x')}}$$

We can solve this equation through two substitutions and using an integral table for one of integrals, [6]. Then we get the following solutions for $B_x$ component:

$$\begin{aligned}B_x = &-p(R_2,\theta)\Big[S(\theta,R_2,z_2,x_{2R})(\mathbf{r}) - S(\theta,R_2,z_2,x_2)(\mathbf{r}) - S(\theta,R_2,z_1,x_{2R})(\mathbf{r}) + S(\theta,R_2,z_1,x_2)(\mathbf{r})\Big]\\ &+ p(R_1,\theta)\Big[S(\theta,R_1,z_2,x_{1R})(\mathbf{r}) - S(\theta,R_1,z_2,x_1)(\mathbf{r}) - S(\theta,R_1,z_1,x_{1R})(\mathbf{r}) + S(\theta,R_1,z_1,x_1)(\mathbf{r})\Big]\\ &+ |\sin\theta|\Big[D(x_{2R},\theta,R_2,z_1,z_2) + D(x_2,\theta,R_2,z_2,z_1) + D(x_{1R},\theta,R_1,z_2,z_1) + D(x_1,\theta,R_1,z_1,z_2)\Big]\end{aligned}$$

Where:

$$p(R,\theta) = \begin{cases} \dfrac{|\sin\theta|(x+\cos\theta(y\sin\theta - R) - x\sin^2\theta)}{\sqrt{x^2\sin^2\theta + (y\sin\theta - R)^2 - (\cos\theta(y\sin\theta-R) - x\sin^2\theta)^2}} & \text{If } y\sin\theta - R \neq 0,\\ |\cos\theta| & \text{If } y\sin\theta - R = 0\end{cases}$$

$$S(\theta,R,z_b,x')(\mathbf{r}) = -2M_0\arctan\left(\frac{u(\theta,R,x') + \sqrt{q^2(\theta,R,z_b) + u^2(\theta,R,x')} + (z-z_b)|\sin\theta|}{l(\theta,R)}\right)$$

$$D(x_b,\theta,R,z_r,z_l) = \begin{cases} M_0\ln\left(\dfrac{z-z_l}{z-z_r}\right) & \text{If } \sin^2\theta = 1 \wedge y\sin\theta - R = 0 \wedge x_b - x = 0,\\ M_0\ln\left(\dfrac{D_t(x_b,\theta,R,z_r)}{D_t(x_b,\theta,R,z_l)}\right) & \text{Elsewhere}\end{cases}$$

$$D_t(x_b,\theta,R,z_b) = \sqrt{q^2(\theta,R,z_b) + u^2(\theta,R,x_b)} + (z-z_b)|\sin\theta|$$

$$u(\theta,R,x') = x' + \cos\theta(y\sin\theta - R) - x\sin^2\theta$$

$$h^2(\theta,R) = x^2 + \left(y - \frac{R}{\sin\theta}\right)^2 - \left(\cos\theta\left(y - \frac{R}{\sin\theta}\right) - x\sin\theta\right)^2$$

Then:

$$q^2(\theta,R,z_b) = \left[(z-z_b)^2 + x^2\right]\sin^2\theta + (y\sin\theta - R)^2 - \left[\cos\theta(y\sin\theta - R) - x\sin^2\theta\right]^2$$

$$l(\theta,R) = |\sin\theta|h(\theta,R) = \sqrt{x^2\sin^2\theta + (y\sin\theta - R)^2 - \left[\cos\theta(y\sin\theta-R) - x\sin^2\theta\right]^2}$$

### $B_y$ component

For $B_y$ component, we have:

$$\begin{aligned}B_y = &\, j(R_2,\theta)\Big[-S(\theta,R_2,z_2,x_{2R})(\mathbf{r}) + S(\theta,R_2,z_2,x_2)(\mathbf{r}) + S(\theta,R_2,z_1,x_{2R})(\mathbf{r}) - S(\theta,R_2,z_1,x_2)(\mathbf{r})\Big] +\\ &j(R_1,\theta)\Big[S(\theta,R_1,z_2,x_{1R})(\mathbf{r}) - S(\theta,R_1,z_2,x_1)(\mathbf{r}) - S(\theta,R_1,z_1,x_{1R})(\mathbf{r}) + S(\theta,R_1,z_1,x_1)(\mathbf{r})\Big] +\\ &\frac{|\sin\theta|}{\sin\theta}\cos\theta\Big[D(x_1,\theta,R_1,z_2,z_1) + D(x_{1R},\theta,R_1,z_1,z_2) + D(x_{2R},\theta,R_2,z_2,z_1) + D(x_2,\theta,R_2,z_1,z_2)\Big]\end{aligned}$$

Where:

$$j(R,\theta) = \begin{cases} \dfrac{y|\sin\theta| - R\dfrac{|\sin\theta|}{\sin\theta} + \Big(x\sin^2\theta - \cos\theta(y\sin\theta - R)\Big)\cos\theta\dfrac{|\sin\theta|}{\sin\theta}}{\sqrt{x^2\sin^2\theta + (y\sin\theta-R)^2 - (\cos\theta(y\sin\theta - R) - x\sin^2\theta)^2}} & \text{If } y\sin\theta - R \neq 0,\\ \dfrac{\sin\theta\cos\theta}{|\cos\theta|} & \text{Otherwise}\end{cases}$$

**$B_z$ component**
For $B_z$ component, the integral is more simple; first we integrate respect to $x'$ variable doing the previous substitutions and then integrate with $z'$:

$$B_z = M_0 \ln\left(\frac{k(\theta,R_2,z_2,x_{R2})k(\theta,R_2,z_1,x_2)k(\theta,R_1,z_1,x_{R1})k(\theta,R_1,z_2,x_1)}{k(\theta,R_2,z_1,x_{R2})k(\theta,R_2,z_2,x_2)k(\theta,R_1,z_2,x_{R1})k(\theta,R_1,z_1,x_1)}\right)$$

Where, $k(\theta,R,z_b,x_b) = \sqrt{q^2(\theta,R,z_b)+u^2(\theta,R,x')}+u(\theta,R,x')$.

**Results for $\theta = 0$ or $\theta = 180°$**
Since these formulas do not work when $\theta = 0$ or $\theta = 180°$, because in this cases we have: $|\mathbf{r}-\mathbf{r}'| = \sqrt{(z-z')^2+(x-R\cos\theta)^2+(y-y')^2}$.

We use the follow notation, $\phi_0(y_l,y_r,\theta,R)(\mathbf{r}) = \displaystyle\int_{z_1}^{z_2}\int_{y_l}^{y_r}\frac{M_0\,dy'dz'}{\sqrt{(z-z')^2+(x-R\cos\theta)^2+(y-y')^2}}$.
therefore, we have:

$$\phi_0(y_l,y_r,\theta,R)(\mathbf{r}) = f_0(y_l,y_r,\theta,R,z_2)(\mathbf{r}) - f_0(y_l,y_r,\theta,R,z_1)(\mathbf{r})$$

Where:

$$f_0(y_l,y_r,\theta,R,z_b)(\mathbf{r}) = -M_0\int_{y_l}^{y_r}\ln\left(\sqrt{(z-z_b)^2+\rho_0^2(\theta,R,y')}+z-z_b\right)dy'$$

**$B_{x0}$ component (with $\theta = 0$)**

$$\begin{aligned}B_{x0} =& -S_0\left(\theta,R_2,z_2,\frac{a}{2}\right)(\mathbf{r}) + S_0\left(\theta,R_2,z_2,-\frac{a}{2}\right)(\mathbf{r}) + S_0\left(\theta,R_2,z_1,\frac{a}{2}\right)(\mathbf{r}) - S_0\left(\theta,R_2,z_1,-\frac{a}{2}\right)(\mathbf{r})\\ &+ S_0\left(\theta,R_1,z_2,\frac{a}{2}\right)(\mathbf{r}) - S_0\left(\theta,R_1,z_2,-\frac{a}{2}\right)(\mathbf{r}) - S_0\left(\theta,R_1,z_1,\frac{a}{2}\right)(\mathbf{r}) + S_0\left(\theta,R_1,z_1,-\frac{a}{2}\right)(\mathbf{r})\end{aligned}$$

Where:

$$S_0(\theta,R,z_b,y_b)(\mathbf{r}) = \begin{cases} 0 & \text{If } x-R\cos\theta = 0,\\ 2M_0\dfrac{x-R\cos\theta}{|x-R\cos\theta|}\arctan\left(\dfrac{y-y_b+\sqrt{q_0^2(\theta,R,z_b)+(y-y_b)^2}+(z-z_b)}{|x-R\cos\theta|}\right) & \text{Otherwise}\end{cases}$$

$$q_0^2(\theta,R,z_b) = (z-z_b)^2+(x-R\cos\theta)^2$$

**$B_{y0}$ component (with $\theta = 0$)**

$$\begin{aligned}B_{y0} =& D_0\left(\theta,R_2,\frac{a}{2},z_1,z_2\right)(\mathbf{r}) + D_0\left(\theta,R_2,-\frac{a}{2},z_2,z_1\right)(\mathbf{r}) + D_0\left(\theta,R_1,\frac{a}{2},z_2,z_1\right)(\mathbf{r})\\ &+ D_0\left(\theta,R_1,-\frac{a}{2},z_1,z_2\right)(\mathbf{r})\end{aligned}$$

Where:

$$D_0(\theta,R,y_b,z_n,z_d)(\mathbf{r}) = \begin{cases} M_0\ln\left(\dfrac{|z-z_d|}{|z-z_n|}\right) & \text{If } x-R\cos\theta = 0 \wedge y-y_b = 0,\\ M_0\ln\left(\dfrac{D_{t0}(\theta,R,y_b,z_n)}{D_{t0}(\theta,R,y_b,z_d)}\right) & \text{otherwise}\end{cases}$$

$B_{z0}$ **component (with** $\theta = 0$**)**

$$B_{z0} = M_0 \ln \left( \frac{k_0\left(\theta, R_2, -\frac{a}{2}, z_2\right) k_0\left(\theta, R_2, \frac{a}{2}, z_1\right) k_0\left(\theta, R_1, \frac{a}{2}, z_2\right) k_0\left(\theta, R_1, -\frac{a}{2}, z_1\right)}{k_0\left(\theta, R_2, \frac{a}{2}, z_2\right) k_0\left(\theta, R_2, -\frac{a}{2}, z_1\right) k_0\left(\theta, R_1, -\frac{a}{2}, z_2\right) k_0\left(\theta, R_1, \frac{a}{2}, z_1\right)} \right)$$

Where:

$$k_0(\theta, R, y_b, z_b) = \sqrt{q_0^2(\theta, R, z_b) + (y - y_b)^2} + y - y_b$$

We present a picture of magnetic field from a cubic magnet of 4 cm length, with a magnetization vector of 518.67G of magnitude (see Figure 2). To confront this results, we take data from Dexter Magnetic Tecnologies [7]. We take a cubic magnet with $\theta = 270°$, $R_1 = 0$ in, $R_2 = 2$ in, $z_1 = -1$ in, $z_2 = 1$ in, and $a = 2$ in; Residual induction (Br) G= 500, Material type, Nd-Fe-B. We use a point $(x_p, y_p, z_p)$ to calibrate the magnetic field magnitude, for example we take $M_0 = 1$ and calculate the magnetic field magnitude in this point, through our formulas, called $B_f(x_p, y_p, z_p)$. Then we get the experimental result called $B_e(x_p, y_p, z_p)$, then $M_0 = B_e(x_p, y_p, z_p)/B_f(x_p, y_p, z_p)$, this way we can get the correct magnetization vector, in this case is $-37.894$G. We put both experimental and simulated results in a Table 1. Therefore these results help us to show the validity of our formulas. Now we present the results for a hexapole system.

**Table 1.** Confront of data.

| Coordinates (in) | | | B (G) Experimental | | | B (G) Simulated | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | $y$ | $z$ | $B_x$ | $B_y$ | $B_z$ | $B_x$ | $B_y$ | $B_z$ |
| 0 | 1 | 0 | 0 | 64.182 | 0 | 0 | 64.18203 | 0 |
| 0 | 2 | 0 | 0 | 21.6 | 0 | 0 | 21.59964 | 0 |
| 0 | 3 | 0 | 0 | 9.352 | 0 | 0 | 9.35169 | 0 |
| 0 | 4 | 0 | 0 | 4.824 | 0 | 0 | 4.82424 | 0 |
| 0 | 10 | 0 | 0 | 0.455 | 0 | 0 | 0.45542 | 0 |
| 0 | 1 | 1 | 0 | 38.530 | 29.885 | 0 | 38.53027 | 29.88534 |
| 0 | 1 | 2 | 0 | 6.546 | 20.439 | 0 | 6.54579 | 20.43920 |
| 0 | 1 | 3 | 0 | -0.621 | 8.948 | 0 | -0.62051 | 8.94804 |
| 0 | 0 | 3 | 0 | -6.803 | 8.258 | 0 | -6.80321 | 8.25755 |
| 0 | -1 | 2 | 0 | -32.091 | 0 | 0 | -32.09102 | 0 |
| 1 | 1 | 1 | 20.59 | 23.287 | 20.59 | 20.59032 | 23.28694 | 20.59032 |
| 2 | 1 | 1 | 15.491 | 3.856 | 7.452 | 15.49089 | 3.85553 | 7.45216 |

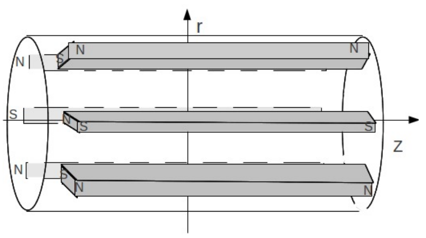


**Table 2.** Physical scheme (hexapolar system).

## 3. Hexapolar system

The physical system consists in a six parallelepiped bar magnets placed on the outside of a cylinder, in alternating North/South polarizations, whose poles are targeted radially; this system is showed in Figure 2.

The cylinder chamber radius is 3.9956cm, with length of 25.9720cm. Longitudinal size magnet bar is 21.0cm, the inner pole radius is $R_1 = 4.1$cm in Figure 2; the external pole radius is $R_2 = 7.1$cm, width $a = 2.5$cm. For hexapole field simulation we take the following angles $\theta = 30°$, $90°$, $150°$, $210°$, $270°$ and $330°$. The field graphic on $XY$ section (plane $z = 0$cm) is showed in Figure 3; where the cusp curvature is seen near to magnetic pole zone. In this figure we can see a gradient radial from the colour scale of field magnitude toward $Z$ axis; this property is necessary to press the plasma back inside camera discharge.

The field on $z = 10.0$cm plane (See Figure 4) is like one central configuration since it is inside region covered by hexapole system. The field on $z = 12.5$cm plane (Figure 5) is more longitudinal, specially in regions near magnet poles; since this zone is outside of hexapole region. The longitudinal field view on $x = 0$cm plane, is given in Figure 6, the magnetic field becomes more longitudinal, outside of region covered by magnets, in the zone near to the magnets.

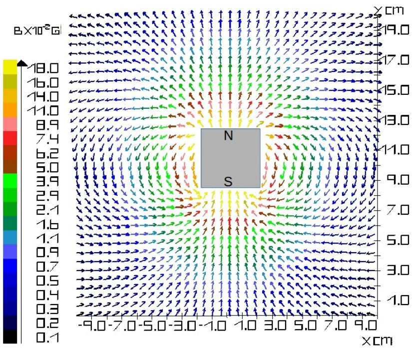


**Figure 2.** Physical system.

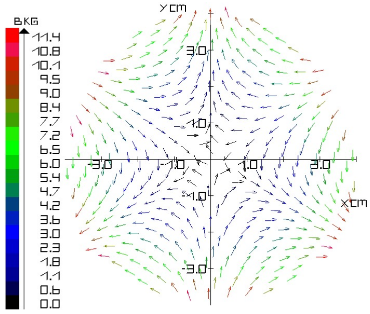


**Figure 3.** XY view (plane $z = 0.0$cm) of the magnetic field.

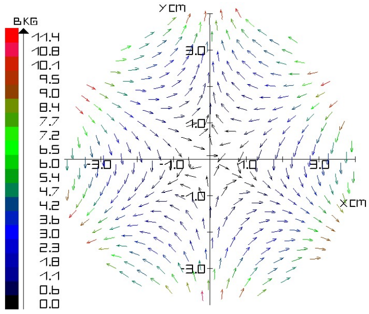


**Figure 4.** XY view (plane $z = 10.0$cm) of the magnetic field.

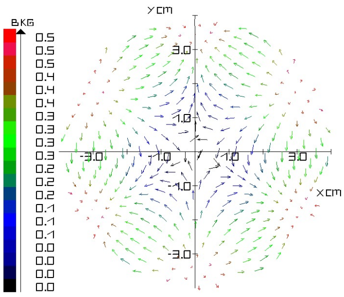


**Figure 5.** XY view (plane $z = 12.5$cm) of the magnetic field.

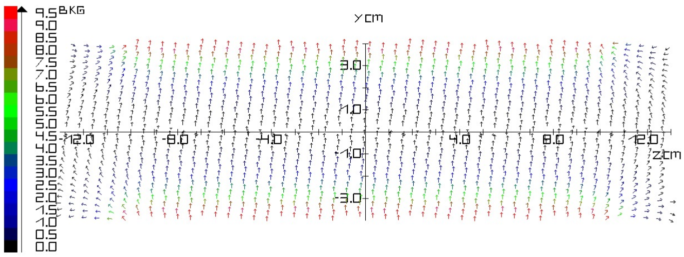


**Figure 6.** YZ view (plane $x = 0.0$cm) of the magnetic field.

## 4. Conclusion

The analytic calculation from a parallelepiped magnet, which can turn around an axis is very important to researchers on plasma physics under multipolar magnetic field, for example, in plasma magnetic confinement in ECR sources, since hexapole magnetic field configurations are an important term to improve the plasma dynamics simulations. Using an approximate value of magnetic field can cause mistakes. Although these errors can be small, the plasma simulations cycles need to be performed many times to reach the stability of the system. Therefore the errors can increase when the number cycles grow. These calculations serve to help reduce computational effects over the plasma behaviour simulated.

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