Physics Letters A 375 (2011) 3655–3658

Contents lists available at SciVerse ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

# Chaotic motion in axially symmetric potentials with oblate quadrupole deformation

Patricio S. Letelier [a,1], Javier Ramos-Caro [a,*], Framsol López-Suspes [b]

[a] Departamento de Matemática Aplicada, IMECC, Universidade Estadual de Campinas, 13083-859, Campinas, SP, Brazil
[b] Facultad de Telecomunicaciones, Universidad Santo Tomás and Escuela de Física, Universidad Industrial de Santander, Bucaramanga, Colombia

A R T I C L E I N F O

*Article history:*
Received 13 July 2011
Received in revised form 22 August 2011
Accepted 23 August 2011
Available online 30 August 2011
Communicated by P.R. Holland

*Keywords:*
Chaos
Gravitation
Nuclear physics
Atomic physics
Mean field approach

A B S T R A C T

By computing the Poincaré's surfaces of section and Lyapunov exponents, we study the effect of introducing an oblate quadrupole in the dynamics associated with two generic spherical potentials of physical interest: the central monopole and the isotropic harmonic oscillator. In the former case we find saddle points in the effective potential, in contrast to the statements presented by Guéron and Letelier in [E. Guéron, P.S. Letelier, Phys. Rev. E 63 (2001) 035201]. The results we show in the second case have application in nuclear or atomic physics. In particular, we find values of oblate deformation leading to a disappearance of shell structure in the single-particle spectrum.



## 1. Introduction

The motion of test particles in force fields with oblate deformations has been a topic of broad interest in different branches of physics, ranging from quantum mechanics to astrophysics. For more than two hundred years this problem has been analyzed in astronomy, motivated by the wide variety of astrophysical objects that can be represented as deformed centers of attraction endowed with axial symmetry. The Sun, the solar system's planets and many galaxies are important examples [1,2]. In the quantum realm it has been reported a variety of deformed nuclei and metallic clusters with axial symmetry (see for example [3–6] and references therein) and the analysis of classical orbits in the mean field of such structures has proven to be fruitful [7–9].

Chaos and regularity associated with potentials modeled by an inverse square law plus a quadrupole-like term, is a topical question studied in recent decades. In Refs. [10–12] the basis of the corresponding homoclinic phenomena has been established, for the case of oblate deformation. In Ref. [13] it is shown that the motion around prolate deformed centers of attraction is chaotic, in contrast to the case of oblate deformation, where no numerical evidences of irregular orbits are found.

In parallel, similar studies have been conducted in atomic and nuclear physics on the basis of the mean field approach, a widely used scheme in the many-body quantum theory. It has been found that orbits around axially symmetric nuclei with oblate (or prolate) deformation are regular and they can exhibit chaotic behavior only when additional multipolar terms are considered [8,9,14].

In this Letter we shall clarify that the assumption about the integrability (or regularity) of motion in oblate deformed potentials, is not entirely true and it has a certain validity range, both in the context of gravitational (or electrostatic) phenomena, as well as nuclear (or atomic) physics. In the former, previous studies of the homoclinic phenomena provide an insight to verify the existence, or not, of chaotic motion. Indeed we shall show numerical evidence of bounded irregular orbits for the aforementioned case. In the second case, the interaction with an oblate core is modeled by a potential that, in cylindrical coordinates $(R, z, \varphi)$, is proportional to $R^2 + z^2/b^2$, where $b < 1$ (it also has extensively been used in modeling metallic cluster deformations [5]). Evidently, the motion in such mean field potential is completely integrable. Yet another would be the case if the mean field potential were modeled as the contribution of a spherical term proportional to $R^2 + z^2$ plus an oblate quadrupole. Although both models would describe adequately the field force, with almost the same precision, the second introduces nontrivial changes in the behavior of orbits. In the second model the Hamiltonian can be split as $H_o + H'$, where $H_o$ and $H'$ are an integrable Hamiltonian and a (small) perturbation, respectively. Since $H'$ is nonseparable in the spatial coordinates, presumably it could introduce some *chaos* in the motion of test

* Corresponding author.
*E-mail addresses:* javier@ime.unicamp.br (J. Ramos-Caro), framsol@gmail.com (F. López-Suspes).

[1] R.I.P.

0375-9601/$ – see front matter 
doi:10.1016/j.physleta.2011.08.050

particles. In fact, here we show that the stochastic regions in phase space can become significant, for modest values of the quadrupolar moment.

A preliminary indicator for chaos is the presence of saddle points in the potential that determines the motion of test particles (i.e. the effective potential) [14]. In the cases studied here, the quadrupolar contribution introduces a saddle point in the equatorial plane. For the gravitational (electrostatic) effective potential it is possible to compute them analytically, contrary to the statement made in Ref. [13]. In both cases we find situations characterized by a dominantly regular structure of the phase space, with chaotic regions of relatively small size. Perhaps this fact has hampered the discovery of numerical evidence for chaotic orbits in the former case [13]. In the second case we can find more prominent stochastic zones in the phase space, which introduces new implications in a further quantum-mechanical approach, as we shall show below.

## 2. Motion in central potentials with oblate deformation

Consider a test particle under the action of an axially symmetric force field described by a spherical term $U_o$ and a quadrupolar contribution

$$U(R,z) = U_o(R,z) - \frac{\beta(2z^2 - R^2)}{2(R^2+z^2)^{\frac{5}{2}}}, \tag{1}$$

where $\beta$ is the quadrupolar moment. It usually represents the major deviation from the spherical symmetry. The case $\beta < 0$, on which we focus here, corresponds to bodies with oblate deformation ($\beta > 0$ represents prolate deformation). For the spherical term, we consider two generic forms corresponding to situations of physical interest: (a) $U_o = -\alpha/\sqrt{R^2+z^2}$, when we are dealing with a gravitational (electrostatic) problem, and (b) $U_o = \alpha(R^2+z^2)$ in the case of the spherical shell model associated with a quantum many-body system (sometimes this term is also used to model the mean gravitational field in galactic dynamics). In the former case $\alpha$ is the monopolar term, which is proportional to the central body mass (charge) and, in the latter case, $\alpha$ is proportional to the quadratic frequency of the three-dimensional isotropic harmonic oscillator (IHO). In both cases the parameter $\alpha$ is a positive real constant. We shall use $\alpha = 1$ without loss of generality.

The orbital motion, in cylindrical coordinates $(R, z, \varphi)$, is given by the set of equations $\dot{R} = v_R$, $\dot{z} = v_z$, $\dot{v}_R = -\tilde{U}_{,R}$, $\dot{v}_z = -\tilde{U}_{,z}$, along with the conservation of the *energy*, $E = (v_R^2 + v_z^2)/2 + \tilde{U}$. Here, $\tilde{U} = U + \ell^2/2R^2$ is the effective potential with $\ell$ being the $z$-component of the angular momentum of the test particle. Whenever $\tilde{U} \leqslant E$ holds, the motion is bounded to a three-dimensional surface in the $(R, z, v_R, v_z)$ phase space. This fact enables us to use the method of Poincaré's surface of section to provide a qualitative description of regularity or chaoticity of orbits. The apparition of stochastic zones in the surfaces of section is due to the existence of saddle points in $\tilde{U}$. In order to find them, we have to determine the stationary points of the dynamical system, which are precisely the critical values of $\tilde{U}$. They are found by solving the coupled equations $\tilde{U}_{,R}(R,z) = 0$ and $\tilde{U}_{,z}(R,z) = 0$, for $R$ and $z$. Since $\tilde{U}$ has symmetry of reflection with respect to the equatorial plane, the second equation is fulfilled at $z = 0$. We start by analyzing the situation (a), which we shall denote as *monopole + oblate quadrupole* (M + OQ). Then we shall focus on the case (b), denoted as IHO + OQ.

### 2.1. (a) Monopole + oblate quadrupole

In this case we can find the radial coordinates of the critical points analytically:

$$R_\pm = \left(\ell^2 \pm \sqrt{\ell^4 + 6\alpha\beta}\right)/(2\alpha). \tag{2}$$

The critical values correspond to saddle points if the Hessian (Gaussian curvature of $\tilde{U}$), $\Delta = \tilde{U}_{,RR}\tilde{U}_{,zz} - \tilde{U}_{,Rz}^2$, evaluated at $(R_\pm, 0)$, is a negative real number. We have that

$$\Delta_\pm = \left(2\alpha R_\pm^2 - 9\beta\right)\left[6\beta + R_\pm\left(3\ell^2 - 2\alpha R_\pm\right)\right]/\left(2R_\pm^{10}\right). \tag{3}$$

We note that, for $\beta < 0$ the Hessian satisfies $\Delta_+ > 0$ and $\Delta_- < 0$ if and only if $\ell^4 > 6\alpha|\beta|$, which is the necessary condition that $R_+$ and $R_-$ be real positive numbers. This means that whenever $\tilde{U}$ has critical points in the equatorial plane, one of them is a saddle point. In consequence, if the orbital energy satisfies the condition

$$E > -\frac{2\alpha^2\ell^2(\ell^2 - \sqrt{\ell^4 + 6\alpha\beta}) + 8\alpha^3\beta}{(\ell^2 - \sqrt{\ell^4 + 6\alpha\beta})^3} \tag{4}$$

it is probable to find chaotic orbits near the point $(R_-, 0)$, i.e. the homoclinic circular orbit [12]. In the case of an attractive field force, we have a bounded motion if $E \leqslant 0$, in addition to (4). This fact leads to the additional condition $\ell^4 < 8\alpha|\beta|$. Thus we get

$$0.75 < \frac{\ell^4}{8\alpha|\beta|} < 1, \tag{5}$$

which can be regarded as the prerequisite to find bounded chaotic motion when we are dealing with attractive potentials modeled as a monopolar term plus an oblate deformation.

In order to illustrate the above statements, we show in Fig. 1 the Poincaré surface of section generated by orbits around a body with quadrupolar moment $\beta = -0.057$, corresponding to $E = -0.32$ and $\ell = 0.8$ (this set of values, given in dimensionless units, fulfills the relations (4) and (5)). In this case we find a chaotic orbit with initial radius $R = 0.198$ (and $z = 0$), which is very close to the radial coordinate of the saddle point, $R_- \approx 0.19$. Note that the stochastic region is very small, in relation to the prominent regular zones that dominate the phase space structure (Fig. 1(a)). Fig. 1(b) shows a detail of the chaotic region, which contains a substructure of regular islands. In order to quantify the degree of regularity of these orbits, we estimate the associated largest Lyapunov exponents, represented here by $\lambda$, in Table 1. We find that the chaotic orbit is characterized by $\lambda \approx 0.02$, while for the other cases it is about $10^{-4}$, which presumably indicates regularity ($\lambda \to 0$).

The above physical parameters can be used consistently at mesoscopic or atomic scales. A surface of section, as the above mentioned, would provide a description for the classical orbit of an electron with energy $\approx 5$ MeV interacting with a nucleus with mass number $\approx 200$ and quadrupolar electric moment $\approx -5$ electron-barns. In such case the saddle point would be at a distance $\approx 2 \times 10^{-15}$ m from the nucleus. At astronomical scales there are some restrictions. In celestial mechanics the quadrupolar moment is given by $\beta = \alpha J_2 R_o^2$ where $J_2$ and $R_o$ are the distortion factor and the equatorial radius of the source, respectively [1]. According to relations (2) and (5) we can determine a range of locations of saddle points such that it is possible to find bounded unstable orbits: $0.707 < |J_2|^{-1/2} R_-/R_o < 1.225$. For example, in the case of the Earth, $J_2 = -1.082 \times 10^{-3}$ and $R_o = 6378$ km, and the maximum allowed value for $R_-$ is about 3.2 km, which does not belong to the physical domain of the quadrupolar approximation. However, the non-integrability of orbits is physically meaningful for objects with larger oblate distortion, such as galaxies. Indeed, we can find chaotic orbits belonging to the physical domain of (1) for a compact body with distortion parameter $|J_2| > 2/3$, and also for a flat ring with a central point mass (a black hole, for example), if the interior radius is relatively far from the center.

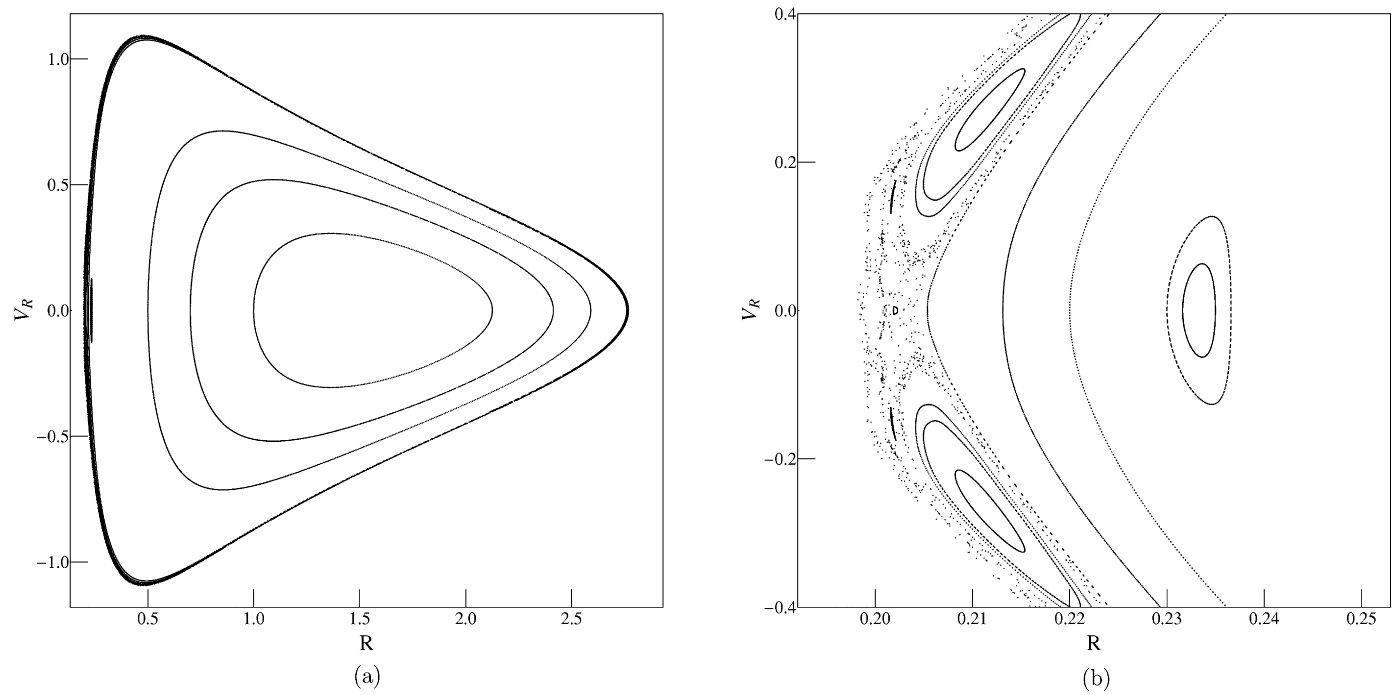


**Fig. 1.** (a) Surface of section for orbits with $E = -0.32$, $\ell = 0.8$ around a body with quadrupolar moment $\beta = -0.057$. (b) Detail of the figure (a), showing a chaotic region enclosing regularity islands.

**Table 1**
Estimation of the largest Lyapunov exponent, λ, corresponding to orbits with initial conditions $R = R_o$, $z = 0$, $V_R = 0$ which are drawn in the surfaces of section of Figs. 1 (left) and 2 (right).

| M + OQ | | IHO + OQ | |
|---|---|---|---|
| $R_o$ | $10^{-4}\lambda$ | $R_o$ | $10^{-4}\lambda$ |
| 0.198 | $213 \pm 12$ | 0.35 | $473 \pm 25$ |
| 0.213 | $3.48 \pm 0.22$ | 0.60 | $1.48 \pm 0.22$ |
| 0.220 | $2.31 \pm 0.34$ | 0.80 | $2.31 \pm 0.34$ |
| 0.230 | $1.86 \pm 0.13$ | 0.90 | $1.45 \pm 0.13$ |
| 0.500 | $1.73 \pm 0.23$ | 1.00 | $1.80 \pm 0.23$ |
| 0.700 | $1.32 \pm 0.17$ | 1.20 | $1.27 \pm 0.17$ |
| 1.000 | $1.19 \pm 0.25$ | 1.40 | $1.05 \pm 0.25$ |

### *2.2. (b) Isotropic harmonic oscillator + oblate quadrupole*

In this case the radial location of the equilibrium points, $R_c$, is given by $\ell_c^2 = (4\alpha R_c^5 - 3\beta)/2R_c$, where $\ell_c$ is the angular momentum of the equatorial circular orbit with radius $R_c$. The Hessian is

$$\Delta = 4\alpha^2\left(1 - \frac{9\beta}{4\alpha R_c^5}\right)\left(4 + \frac{3\beta}{4\alpha R_c^5}\right), \tag{6}$$

from which we see that $\Delta < 0$, only if $R_c^5 < 3|\beta|/16\alpha$ (of course, we are assuming $\beta < 0$ and $\alpha > 0$). This upper limit is precisely the critical value at which $\ell_c^2$ reaches the minimum and, in consequence, we can establish that

$$\chi \equiv \frac{\ell^2}{\alpha^{1/5}|\beta|^{4/5}} > 2.6206 \tag{7}$$

is now the relation that provides the prerequisite for the existence of chaotic motion. Of course, we have bounded orbits if the test particle's energy is greater than the energy of the circular orbit, $E_c = (8\alpha R_c^5 - \beta)/4R_c^3$. In Fig. 2(a) we plot the surface of section corresponding to $\beta = -0.1$ and $\ell = 0.8$, with an energy $E = 3.2$. We find a chaotic orbit with initial conditions $R = 0.35$, $z = 0$, $V_R = 0$. Similar to the case of potential (a), the stochastic region is small in relation with the prominent zone of KAM curves. However, the chaotic motion is now discernible in a larger region of phase space. In Fig. 2(b) we show a detail of the stochastic zone with a structure very similar to Fig. 1(b). The associated λ is shown in Table 1 (right).

The results reported in this case can be relevant for further quantum-mechanical analysis of many-body systems, like nuclei and metallic clusters. It is known that invariant tori of the Poincaré sections are close related to shell structures in the single-particle spectrum [9]. Accordingly, if we find chaos in the classical problem, a disappearance of shell structure should be expected in the analogous quantum case [15].

As a check, surfaces of section and λ were computed by solving the equations of motion by two methods: (i) the fourth order Runge–Kutta method with variable time step and incorporating the Hénon's trick [17]; (ii) symplectic Runge–Kutta algorithms of 4th, 5th and 6th order. For a time of integration $t \sim 10^4$ we do not find significative differences in the results obtained, except for the maximum cumulative error in the energy conservation: $10^{-12}$ and $10^{-15}$ for (i) and (ii), respectively. Calculations were performed in the LCP (Laboratório de Computação Paralela) at IMECC.

The value for the above integration time is reported in dimensionless units, so it could be more useful expressing $t$ in a system of units depending on the situation we are dealing. As an example,

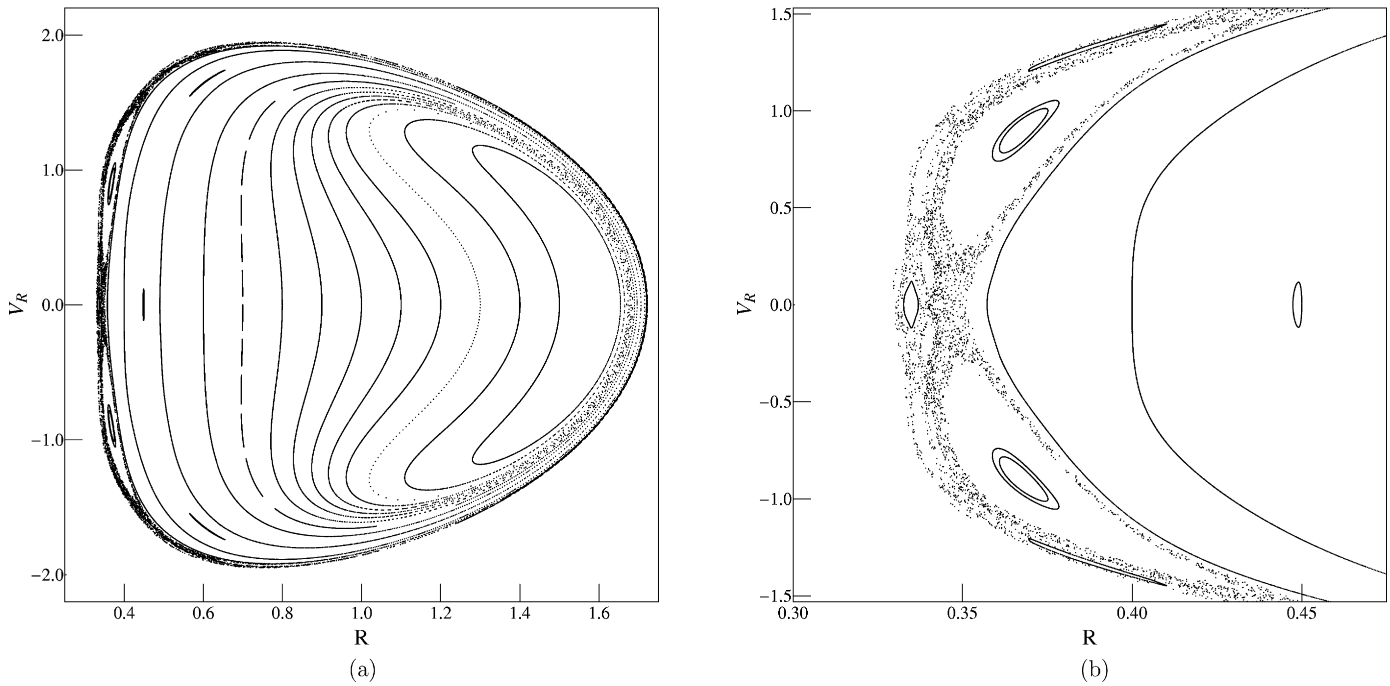


**Fig. 2.** (a) Surface of section for some orbits with $E = 3.2$ and $\ell = 0.8$ in the harmonic oscillator potential with oblate deformation $\beta = -0.1$. (b) Detail of the figure (a), showing a prominent chaotic region with resonant islands inside.

for the case of a galaxy with $10^{10}$ solar masses and equatorial radius $R_o \sim 10$ kpc, where the typical velocity of stars is about 100 km/s, we have $t \sim 10^{12}$ yr.

## 3. Concluding remarks

Contrary to the statements of Guéron and Letelier [13], we find numerical evidences of chaotic motion around a monopole plus oblate quadrupole. Those authors pointed out that the saddle points are given by $R = \sqrt{(3\beta - 2\ell^2)/2\alpha}$, $z = 0$ along with the conditions $\ell^2 < 3\beta$ and $3\ell^2 > \sqrt{2\alpha/3}$. In consequence, they stated that irregular orbits were possible only with prolate deformation, $\beta > 0$. However, we show here that this is not true, by providing the corrected expressions, i.e. Eqs. (2) and (5), which enable the existence of chaotic orbits around a M + OQ system. The surfaces of section reveal the existence of stochastic zones (in the phase space) of very small size, which hampered their discovery. We point out that the irregularity of motion around M + OQ is meaningful for bodies with $|J_2| > 2/3$, as axisymmetric galaxies, for example.

We also considered the case corresponding to the interaction of a nucleus (atom) with an outer test particle. It is commonly modeled through a spheroidal potential (SP) and the chaotic motion is only possible by the introduction of additional multipole terms (octupole, hexadecapole, etc.) [8,9]. However we presented here an alternative model, denoted as IHO + OQ, which admits chaotic orbits without considering extra multipole terms. This fact suggests that a further quantum-mechanical study might reveal a nontrivial modification in the description of the single-particle spectrum, in relation to the results obtained with the SP. While in the SP case there is shell structure for any oblate (or prolate) deformation, in the IHO + OQ model there are restrictions: According to (7), we would expect oblate shell structure if $0 \leqslant \chi \leqslant 2.6206$ (the parameter $\chi$ is defined in Eq. (7)). In the prolate case, $\beta > 0$, the condition to have saddle points is $0 \leqslant \chi \leqslant 5.1017$, which means that we would expect prolate shell structure for $\chi > 5.1017$. Note that there is a range of instability between the oblate and prolate regimes, $2.6206 < \chi < 5.1017$. It would be interesting to investigate if there exists any connection between such range and the phenomenon of transition from prolate deformations to oblate ones, at certain values of angular momentum [16].

## Acknowledgements

Patricio S. Letelier (R.I.P.) thanked the partial financial support of FAPESP and CNPq. J.R.-C. thanks the financial support from FAPESP. F.L.-S. wants to thank the financial support from *Vicerrectoría Académica*, Universidad Industrial de Santander.

## References

[1] D. Boccaletti, G. Pucacco, Theory of Orbits, vol. 1, third ed., Springer, 2004.
[2] J. Binney, S. Tremaine, Galactic Dynamics, 2nd ed., Princeton University Press, 2008.
[3] J.H. Gundlach, K.A. Snover, J.A. Behr, C.A. Gossett, M. Kicinskahabior, K.T. Lesko, Phys. Rev. Lett. 65 (20) (1990) 2523.
[4] S. Åberg, H. Flocard, W. Nazarewicz, Ann. Rev. Nucl. Phys. A 545 (1992) 589.
[5] S.G. Nilsson, C.F. Tsang, A. Sobiczewsky, Z. Szymanski, S. Wycech, C. Gustafson, I.-L. Lamm, P. Möller, B. Nilsson, Nucl. Phys. A 131 (1969) 1.
[6] M. Brack, Rev. Mod. Phys. 65 (1993) 677.
[7] R. Arvieu, F. Brut, J. Carbonell, J. Touchard, Phys. Rev. A 35 (1987) 2389.
[8] W.D. Heiss, R.G. Nazmitdinov, S. Radu, Phys. Rev. Lett. 72 (15) (1994) 2351.
[9] W.D. Heiss, R.G. Nazmitdinov, S. Radu, Phys. Rev. C 52 (6) (1995) 3032.
[10] M. Irigoyen, C. Simó, Celestial Mech. Dynam. Astronom. 55 (3) (1993) 281.
[11] A. Celleti, P. Negrini, Celestial Mech. Dynam. Astronom. 61 (3) (1995) 253.
[12] P.S. Letelier, W.M. Vieira, Phys. Lett. A 242 (1998) 7.
[13] E. Guéron, P.S. Letelier, Phys. Rev. E 63 (2001) 035201.
[14] L. Junquing, Phys. Rev. Lett. 79 (12) (1997) 2387.
[15] W.D. Heiss, R.G. Nazmitdinov, S. Radu, Phys. Rev. C 52 (3) (1995) 1179.
[16] Y. Sun, P.M. Walker, F.-R. Xu, Y.-X. Liu, Phys. Lett. B 659 (2008) 165.
[17] M. Hénon, Physica D 5 (1982) 412.